# Examples of the Reflective Algebra for Various Hopf Algebras

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# Outline

# Research Motivation

**Hopf algebras**: Important objects which give a generalization of a group in category theory (monoidal categories).

Special type of monoidal category: **Braided** monoidal category.

The study of these categories has numerous applications:

- Knot theory
- Quantum field theory
- String theory

# Research Motivation

One step further: **Module categories** over monoidal categories. They also can be braided.

- Classical objects: (quasitriangular) Hopf algebras, (braided) monoidal categories.

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One step further: **Module categories** over monoidal categories. They also can be braided.

- Classical objects: (quasitriangular) Hopf algebras, (braided) monoidal categories.
- New objects: **Reflective algebras** over quasitriangular Hopf algebras, (braided) module categories over (braided) monoidal categories.

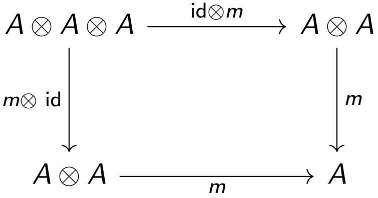
# Research Motivation

**Research Question**

What is the structure of these reflective algebras? Can we classify them?

# Hopf Algebras

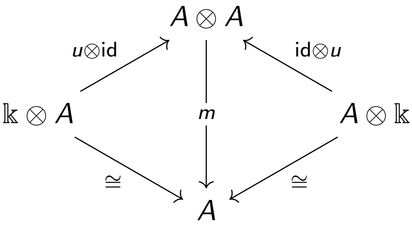
A $\mathbb{k}$-**algebra** is a $\mathbb{k}$ vector space $A$ with linear maps $m : A \otimes A \to A$ and $u : \mathbb{k} \to A$ such that this diagram commutes:




In other words

$$m(a \otimes m(b \otimes c)) = m(m(a \otimes b) \otimes c) \implies a \cdot (b \cdot c) = (a \cdot b) \cdot c.$$

# Hopf Algebras



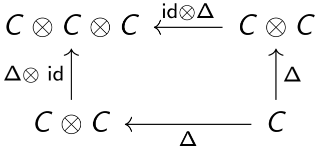

In other words, we have

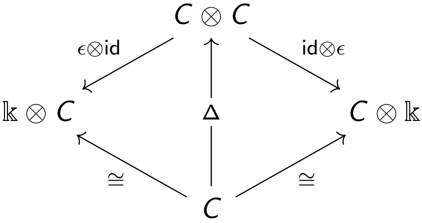
$$m(u(1_{\Bbbk}) \otimes a) = m(a \otimes u(1_{\Bbbk})) = a. \implies u(1_{\Bbbk}) \cdot a = a \cdot u(1_{\Bbbk}) = a.$$

# Hopf Algebras

If we reverse all the arrows in our algebra diagrams, we get the notion of a coalgebra.

A **k-coalgebra** $C$ has linear maps $\Delta : C \to C \otimes C$ and $\epsilon : C \to \mathbb{k}$ such that these diagrams commute:



# Hopf Algebras

A Hopf Algebra is both an algebra and a coalgebra, making it into a **bialgebra**.

The multiplication/comultiplication need to be compatible with one another:

$$\Delta(a \cdot b) = \Delta(a)\Delta(b)$$
$$\epsilon(a \cdot b) = \epsilon(a)\epsilon(b)$$

Finally, a Hopf algebra has an object that acts as an inverse, called the **antipode**.

# Hopf Algebras

Define the **convolution** $*$ for $f, g : A \to A$.

$$A \xrightarrow{\Delta} A \otimes A \xrightarrow{f \otimes g} A \otimes A \xrightarrow{m} A$$

$$A \xrightarrow{f * g} A$$

The set of maps $A \to A$ is an associative algebra with multiplication $*$ and multiplicative identity $u\epsilon$, where $u$ is the unit and $\epsilon$ is the counit.

# Hopf Algebras

The **antipode** $S$ is an element in the set of maps $A \to A$ such that

$$S * \mathrm{id} = \mathrm{id} * S = u\epsilon.$$

# Hopf Algebras

## Example

The group algebra $H = kG$ is a Hopf algebra with:

- Comultiplication $\Delta(g) = g \otimes g$ for $g \in G$
- Counit being the unit in $G$
- Antipode $S(g) = g^{-1}$ for $g \in G$.

## Example

The dual of the group algebra $H = (kG)^*$ is a Hopf algebra with:

- Comultiplication $\Delta(g) = \sum_{g_1, g_2 \in G, g_1 g_2 = g} \delta_{g_1} \otimes \delta_{g_2}$ for $g \in G$
- Counit $\epsilon(\delta_g) = \langle \delta_g, 1 \rangle$ where $g \in G$ and 1 is the unit in $G$.
- Antipode $S(\delta_g) = \delta_{g^{-1}}$ for $g \in G$.

A **category** consists of objects and morphisms. Objects can be anything (sets, topological spaces, groups, etc.) and morphisms are mappings between objects.

Each object has an identity morphism and morphisms are associative. Namely:

1. If we have morphisms $f : A \to B, g : B \to C, h : C \to D$, then $(hg)f = h(gf)$
2. The identity morphism is the morphism $A \to A$. If we have identity morphism $I_A : A \to A$ and morphism $f : A \to B$, then $I_A f = f I_A = f$.

# Introduction to Categories

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**Example**

The category of vector spaces over a field $k$, where the objects are $k$-vector spaces and the morphisms are homomorphisms. This is denoted as $\text{Vect}_k$.

# Monoidal Categories

It turns out that when $H$ is a Hopf algebra, the category of all $H$-modules is a **monoidal** category $\mathcal{C}$

This category is equipped with:

1. A tensor product operation $\otimes$
2. An associativity isomorphism $a$

$$a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z),$$

3. Unit isomorphisms $l_X : e \otimes X \to X$ and $r_X : X \otimes e \to X$.

A monoidal category is intended to be a categorification of a monoid.

# Braided Monoidal Categories

A **braided monoidal category** is a monoidal category $C$ equipped with a commutativity constraint called a *braiding*, which is a natural isomorphism $c_{X,Y} : X \otimes Y \to Y \otimes X$ that satisfies hexagonal axioms.

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A **braided monoidal category** is a monoidal category $C$ equipped with a commutativity constraint called a *braiding*, which is a natural isomorphism $c_{X,Y} : X \otimes Y \to Y \otimes X$ that satisfies hexagonal axioms.

- Quasitriangular Hopf algebras give rise to braided monoidal categories.
- The **Drinfeld double** $D(H) = H \otimes H^{*\mathrm{op}}$ of a finite dimensional Hopf algebra $H$ is a quasitriangular Hopf algebra.

# Module Categories

A **left module category** over a monoidal category $\mathcal{C}$ is a category $\mathcal{M}$ with:

1. An *action* operator $* : \mathcal{C} \times \mathcal{M} \to \mathcal{M}$ satisfying associativity axioms.
2. A unit isomorphism $\lambda_{\_}$ with

$$\lambda_M : e * M \to M$$

for $M \in \mathcal{M}$ and $e \in \mathcal{C}$.

Module categories can also have a braiding. We would like a generalization of the Drinfeld double for module categories.

# The reflective algebra: $R_H(A)$

$R_H(A)$-mod is a left module category over the braided monoidal category $D(H)$-mod. This gives it the structure of a braided module category.

# Results and Goals

**Current Goal**

When $H$ is the Drinfeld double of a Hopf Algebra, and $A = \mathbb{k}$, what is the structure of $R_H(A)$?

# Results and Goals

We have computed this for when $H = D(G)$ with $G$ a group.

A few examples:

- $H = D(C_n)$ with $C_n$ being the cyclic group of order $n$ : Multiplication in $R_H(A)$ is done componentwise.
- $H = D(S_3)$ : $R_H(A)$ consists of 36 elements containing $(\mathbb{k}S_3)^*$ as a subalgebra.

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